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A General Recurrence Interpolation Formula and Its Applications to Multivariate Interpolation

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Following the ideas of several papers by G. Mühlbach, a general recurrence interpolation formula is obtained that contains as particular cases some extended Newton and Aitken–Neville interpolation formulas. The exposition of the problem allows us to show the applications of this formula to multivariate interpolation. This is the principal aim of this work. Some simple examples are given to show the variety of applications.

1. INTRODUCTION

In several papers, Mühlbach ([6–10]) has obtained an interesting generalized Newton formula for the solution of the interpolation problem at given points by functions which form a Chebyshev system, and finally in [11], the same formula is obtained for the general finite linear interpolation problem.

Also in [8] he has extended the Neville–Aitken recurrence formula for the construction of interpolating functions from others which solve easier interpolation problems.

When the present paper was completed the authors learned of two papers that prove Mühlbach's results in two different ways. Håvie [5] uses extrapolation schemes and Brezinski [2] uses a Sylvester's identity on determinants.

But, in the Mühlbach–Neville–Aitken formula, the coefficients λ_i depend on the point x at which the interpolating function is computed. As it involves a product of functions, it was inconvenient to extend the formula to the finite linear interpolation problem.

In this paper we avoid such difficulties by deriving a general formula for the value of a linear form $L(p)$, where p is the solution of the interpolation

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problem. In some particular cases, such as Newton's formula, one can use the formula to obtain directly the solution p of the problem.

In Section 2, we introduce notations and pose the problem. In Sections 3 and 4, we obtain the general formula, study special cases and give some examples. In Section 5, we construct the divided differences by recurrence, applying the results of [11]. Finally, some examples of applications to multivariate interpolation are given. In the cases we have treated this method for the construction of the solution is much slower than the method in [4] if the latter can be applied. But we can use both methods together to enlarge the field of application of [4], as shown in Section 6.

2. Definitions and Notation

Let V be a vector space of dimension $n \in N$ over a commutative field of characteristic zero K.

Let $\{f_1, f_2, ..., f_n\}$ be a subset of V and let

$$\mathscr{L}_n = \{L_1, L_2, ..., L_n\}$$

be a set of linear forms on V.

Obviously one has

$$\det(L_i f_j) \neq 0, \qquad 1 \leqslant i, j \leqslant n, \tag{1}$$

if and only if both sets are linearly independent. Let us assume (1) and let W be the vector space spanned by $\{f_1, f_2, ..., f_m\}$, $m < n$.

For any $z = (z_1, z_2, ..., z_n) \in K^n$ we shall denote by

$$p_n = p_n[z] \tag{2}$$

the unique element in V which verifies

$$L_i p_n = z_i, \qquad i = 1, 2, ..., n. \tag{3}$$

Let $I_{m,i}$, $i = 1, 2, ..., s$ $(1 \leqslant s \leqslant n - m + 1)$, be s subsets of $\{1, 2, ..., n\}$ such that

$$\operatorname{card} I_{m,i} = m, \qquad I_{m,i} \neq I_{m,j}, \quad \forall\, i \neq j.$$

Let us denote

$$\mathscr{L}_{m,i} = \{L_j \mid j \in I_{m,i}\}, \qquad i = 1, 2, ..., s, \tag{4}$$

and assume

$$\det(L_j f_k) \neq 0, \qquad 1 \leqslant k \leqslant m, \quad j \in I_{m,i}. \tag{5}$$

If

$$I_{m,i} = \{j_{i1}, j_{i2}, \ldots, j_{im}\}$$

we shall write

$$z_{m,i} = (z_{j_{i1}}, z_{j_{i2}}, \ldots, z_{j_{im}})$$

and

$$\mathscr{L}_{m,i} f = (L_{j_{i1}} f, \ldots, L_{j_{im}} f).$$

Then, similarly to (2), we shall denote by

$$p_{m,i} = p_{m,i}[z_{m,i}] \tag{6}$$

the unique element of W which verifies

$$L_j p_{m,i} = z_j, \qquad \forall j \in I_{m,i}, \tag{7}$$

and analogously $p_{m,i}^{m+k}$, $k = 1, 2, \ldots, n-m$, such that

$$L_j p_{m,i}^{m+k} = L_j f_{m+k}, \qquad \forall j \in I_{m,i}. \tag{8}$$

Finally we shall write

$$\alpha_n^k = \alpha_n^k[z], \qquad k = 1, 2, \ldots, n, \tag{9}$$

for the coefficient of f_k in (2) and we shall call it the kth divided difference of z with respect to problem (3).

3. A General Recurrence Interpolation Formula

THEOREM 1. *For $s \geqslant 2$, for any linear form L on V such that*

$$D(L) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ Lp_{m,1}^{m+1} & Lp_{m,2}^{m+1} & \cdots & Lp_{m,s}^{m+1} \\ Lp_{m,1}^{m+2} & Lp_{m,2}^{m+2} & \cdots & Lp_{m,s}^{m+2} \\ & & \cdots & \\ Lp_{m,1}^{m+s-1} & Lp_{m,2}^{m+s-1} & \cdots & Lp_{m,s}^{m+s-1} \end{vmatrix} \neq 0 \tag{10}$$

and for any $z \in K^n$, one has

$$Lp_n = \sum_{i=1}^{s} \lambda_i Lp_{m,i} + \sum_{i=1}^{s} \sum_{k=m+s}^{s} \lambda_i \alpha_n^k (Lf_k - Lp_{m,i}^k), \tag{11}$$

where $(\lambda_1, \lambda_2, ..., \lambda_s)$ *is the unique solution of the system*

$$\sum_{i=1}^{s} \lambda_i = 1,$$
$$\sum_{i=1}^{s} \lambda_i L p^j_{m,i} = Lf_j, \qquad j = m+1, ..., m+s-1. \tag{12}$$

For $s = 1$ *we have*

$$p_n = p_{m,1} + \sum_{k=m+1}^{n} \alpha^k_n (f_k - p^k_{m,1}). \tag{13}$$

Proof. Let $s \geqslant 2$. Both members of (11) can be considered as linear forms on K^n applied to z, $L^{(1)}, L^{(2)}$:

$$L^{(1)}z = \text{1st member of (11)},$$
$$L^{(2)}z = \text{2nd member of (11)}.$$

By (1) these linear forms are the same if they take the same value on each $z = \mathscr{L}_n f_j = (L_1 f_j, ..., L_n f_j)$, $j = 1, 2, ..., n$, since these z are a basis of K^n.

Taking $z = \mathscr{L}_n f_j$, $j = 1, 2, ..., m$, in (11) we obtain for the first member Lf_j and

$$\left(\sum_{i=1}^{s} \lambda_i\right) Lf_j$$

for the second one, because for $z = \mathscr{L}_n f_j$, $j = 1, 2, ..., n$,

$$\alpha^k_n = \delta_{jk}. \tag{14}$$

From the first equation in (12) it follows that members coincide.

Now, taking $z = \mathscr{L}_n f_j$, $j = m+1, ..., m+s-1$ in (11) and applying the second equation (12) we get

$$Lf_j = Lf_j.$$

Finally, taking $z = \mathscr{L}_n f_j$, $j = m+s, ..., n$, one has that the first member is Lf_j and the second one is, by (14),

$$\sum_{i=1}^{s} \lambda_i L p^j_{m,i} + \sum_{i=1}^{s} \lambda_i (Lf_j - Lp^j_{m,i}) = Lf_j.$$

For $s = 1$ the same reasoning is true with any L, taking into account that condition (10) disappears and (12) reduces to $\lambda_1 = 1$. Hence we have (13). This formula has been recently obtained by Mühlbach [9] as a generalized Newton formula.

Let us remark that in general one cannot take L off (11) because λ_i are dependent on it.

A particular case is for $s = n - m + 1$. Under the hypothesis (10) the double sum in the second member of (11) disappears and one obtains the generalized Aitken–Neville formula (see [6])

$$Lp_n = \sum_{i=1}^{n-m+1} \lambda_i Lp_{m,i}. \tag{15}$$

We observe, finally, that the case $m = 1$, $s = n - m + 1 = n, \mathscr{L}_{1,i} = \{L_i\}$, $i = 1, 2, \ldots, n$, corresponds to the Lagrange interpolation formula. In this case condition (5) becomes

$$L_i f_1 \neq 0, \qquad \forall i = 1, 2, \ldots, n$$

and (10) can be written

$$\begin{vmatrix} L_1 f_1 & L_2 f_1 & \cdots & L_n f_1 \\ L_1 f_2 & L_2 f_2 & \cdots & L_n f_2 \\ & & \cdots & \\ L_1 f_n & L_2 f_n & \cdots & L_n f_n \end{vmatrix} Lf_1 \neq 0, \tag{16}$$

which, by (1), is equivalent to

$$Lf_1 \neq 0.$$

Then we have

$$Lp_n = \sum_{i=1}^{n} L_i f \cdot Ll_i$$

with

$$L_j l_i = \delta_{ij}.$$

Let us remark that condition (10) can be substituted by other conditions easier to check in application. For example, one has the following result.

THEOREM 2. *Let $\mathscr{L}_{m,i}$, $i = 1, 2, \ldots, s$, $s \geqslant 2$ be such that*

$$\operatorname{card}(\mathscr{L}_{m,i} \cap \mathscr{L}_{m,i-1}) = m - 1, \qquad i = 2, 3, \ldots, s, \tag{17}$$

$$\operatorname{card}\left(\bigcup_{i=1}^{s} \mathscr{L}_{m,i}\right) = m + s - 1, \tag{18}$$

$$\det(L_i f_j) \neq 0, \qquad j = 1, 2, \ldots, m + s - 1; L_i \in \bigcup_{i=1}^{s} \mathscr{L}_{m,i}. \tag{19}$$

If one has, for $i=2,3,\dots,s$,

$$\det(L_i f_j)\neq 0, \qquad j=1,2,\dots,m;\ L_i\in \mathscr{L}_{m,i}\cap\mathscr{L}_{m,i-1}\cup\{L\}, \tag{20}$$

then condition (10) *is verified.*

Proof. By subtracting from each column in (10) the previous column developed by the first row we have

$$D(L)=\begin{vmatrix} Lp_{m,2}^{m+1}-Lp_{m,1}^{m+1} & \cdots & Lp_{m,s}^{m+1}-Lp_{m,s-1}^{m+1} \\ Lp_{m,2}^{m+2}-Lp_{m,1}^{m+2} & \cdots & Lp_{m,s}^{m+2}-Lp_{m,s-1}^{m+2} \\ & \cdots & \\ Lp_{m,2}^{m+s-1}-Lp_{m,1}^{m+s-1} & \cdots & Lp_{m,s}^{m+s-1}-Lp_{m,s-1}^{m+s-1} \end{vmatrix}.$$

If $D(L)=0$ one would have a null linear combination of the rows with coefficients $c_1, c_2,\dots, c_{s-1}$ not all zero.

Calling

$$\varphi=\sum_{i=1}^{s-1} c_i f_{m+i} \tag{21}$$

we should have, by linearity of the interpolation problem,

$$Lp_{m,j}[\varphi]-Lp_{m,j-1}[\varphi]=0, \qquad j=2,3,\dots,s,$$

and since obviously

$$L_i p_{m,j}[\varphi]=L_i p_{m,j-1}[\varphi], \qquad \forall L_i\in\mathscr{L}_{m,j}\cap\mathscr{L}_{m,j-1},$$

one has, by (20),

$$p_{m,j-1}[\varphi]=p_{m,j}[\varphi], \qquad j=2,3,\dots,s.$$

Therefore

$$p_{m,j}[\varphi]=p_{m,k}[\varphi]=\sum_{i=1}^{m}\beta_i f_i, \qquad \forall j,k\in\{1,2,\dots,s\},$$

and

$$\begin{aligned} L_j\varphi &= L_j\left(\sum_{i=1}^{s-1} c_i f_{m+i}\right) \\ &= L_j\left(\sum_{k=1}^{m}\beta_k f_k\right), \qquad \forall L_j\in\bigcup_{i=1}^{s}\mathscr{L}_{m,i}, \end{aligned}$$

which is not possible by (19), except for $c_i=\beta_k=0$, $\forall i,k$.

For the Aitken–Neville formula we remark that by choosing $\mathscr{L}_{m,i}$, $i = 1, 2, \ldots, n-m+1$, such that (17), (18) are verified with $s = n-m+1$ (in this case (19) is the initial hypothesis (1)) and with

$$\mathscr{L}_{m,i} \cap \mathscr{L}_{m,i-1} = \mathscr{L}_{m,j} \cap \mathscr{L}_{m,j-1}, \qquad \forall i, j,$$

we have the idea of the original Aitken's algorithm [1], while if we choose

$$\mathscr{L}_{m,i} = \{L_i, L_{i+1}, \ldots, L_{m+i-1}\}, \qquad i = 1, 2, \ldots, n-m+1,$$

with

$$\mathscr{L}_n = \{L_1, L_2, \ldots, L_n\}$$

we have that of Neville [12].

4. Applications and Examples

In most interpolation problems V is a space of K-valued functions of one or several variables and L and L_i are linear forms representing the value of f and/or some of its derivatives at some point. We shall take them to be so in all our examples. Particularly we shall always assume

$$Lf = f(X), \qquad X \in R^t. \tag{22}$$

There are many examples for which the conditions of Theorem 2 can be easily checked, as we can see in the following example with two variables.

Let $m = 3$, $n = 6$, $s = 4$, $f_1 = 1$, $f_2 = x$, $f_3 = y$, $f_4 = x^2$, $f_5 = xy$, $f_6 = y^2$ and

$$L_i f = f(P_i), \qquad i = 1, 2, \ldots, 6. \tag{23}$$

P_i are distinct points on the plane such that P_4, P_5, P_6 are on a straight line r; P_2, P_3 are on a different line r', whose intersection with r is not any point P_i; and P_1 does not lie either on r or on r'. We know that the problem of finding $p \in \mathscr{P}_2 = \operatorname{span}\{1, x, y, x^2, xy, y^2\}$ such that

$$L_i p = L_i f, \qquad i = 1, 2, \ldots, 6,$$

with L_i defined by (23) has a unique solution, and hence (1) is verified (see, for example [4]).

By taking

$$\mathscr{L}_{3,1} = \{L_1, L_2, L_3\},$$

$$\mathscr{L}_{3,2} = \{L_2, L_3, L_4\},$$

$$\mathscr{L}_{3,3} = \{L_2, L_3, L_5\},$$

$$\mathscr{L}_{3,4} = \{L_2, L_3, L_6\}$$

it is obvious that (17), (18) and (19) are true. Also if the point X does not lie on r', then (20) is true, because

$$\det \begin{bmatrix} 1, x, y \\ P_2, P_3, X \end{bmatrix} \neq 0.$$

By formula (11) we can write the interpolation polynomial of degree two of a given function at $P_1, P_2, ..., P_6$ by means of the four polynomials of degree one which interpolate at four different sets, each with three points.

In general, if we have k points on a straight line r_k, $k-1$ points on another line r_{k-1} (no points on r_k), $k-2$ points on r_{k-2} (no points on r_k, r_{k-1}) and so on, then the corresponding problem is unisolvent in $\mathscr{P}_{k-1}$. If we take the points of the straight lines $r_2, ..., r_{k-1}$ and that of r_1 we shall have a univolvent problem in $\mathscr{P}_{k-2}$, and analogously it will occur with the points of $r_2, r_3, ..., r_{k-1}$ and each one of r_k. Then we can apply Theorem 2 to obtain the solution of the problem in $\mathscr{P}_{k-1}$ from the $k+1$ problems in $\mathscr{P}_{k-2}$ and so on. In this way we obtain an Aitken–Neville algorithm for this problem which is of Aitken's type.

In principle we have to exclude in formula (11) the points X on $r_2, r_3, ..., r_{k-1}$, although this is not an inconvenience by continuity.

Let us see now a simple example where Theorem 2 is not applicable, but Theorem 1 is: $m=4$, $n=6$, $s=2$, $f_1=1$, $f_2=x$, $f_3=y$, $f_4=xy$, $f_5=x^2$, $f_6=x^2y$, with

$$\mathscr{L}_{4,1}f = (f(0,0), f(0,1), f(1,0), f(1,1)), \tag{24}$$

$$\mathscr{L}_{4,2}f = (f(1,0), f(1,1), f(2,0), f(2,1)). \tag{25}$$

Now the condition $D(X) \neq 0$ is $x \neq 1$ and λ_1, λ_2 are

$$\lambda_1 = \frac{3x-2-x^2}{2(x-1)}, \qquad \lambda_2 = \frac{x^2-x}{2(x-1)}.$$

Formula (11) allows us to see the relation between the polynomial in $\mathscr{P}_2$ which interpolates a given function f at the six points (0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1) and the polynomials of degree no greater than 1 in x, y which interpolate f at the involved points in (24) and (25), respectively.

5. Computing the Divided Differences

The divided differences α_n^k, $k=m+s, ..., n$ in (11) can be computed by solving a linear system. If we use Theorem 1 then we shall have an $(n-m)\times(n-m)$ system to obtain $\alpha_n^{m+1}, \alpha_n^{m+2}, ..., \alpha_n^n$, whereas if we use

Theorem 2 the $(n-m-s+1)\times(n-m-s+1)$ system only has the unknowns $\alpha_n^{m+s}, \alpha_n^{m+s+1},..., \alpha_n^n$, which are just those that interest us.

In the first case, by taking any $\mathscr{L}_{m,i}$ and applying Theorem 1 with $s=1$, we have

$$p_n = p_{m,i} + \sum_{k=m+1}^{n} \alpha_n^k (f_k - p_{m,i}^k) \tag{26}$$

and as a particular case of [9, Theorem 1] we get α_n^k, $k=m+1, m+2,..., n$ by solving the linear system

$$\sum_{k=m+1}^{n} \alpha_n^k (L_j f_k - L_j p_{m,i}^k) = z_j - L_j p_{m,i}, \qquad \forall L_j \in \mathscr{L}_n - \mathscr{L}_{m,i}. \tag{27}$$

If we are under the hypothesis of Theorem 2, we have the Newton formula

$$p_n = p\left[\bigcup_{i=1}^{s} z_{m,i}\right] + \sum_{k=m+s}^{n} \alpha_n^k \left(f_k - p\left[\bigcup_{i=1}^{s} \mathscr{L}_{m,i} f_k\right]\right), \tag{28}$$

where $p[\bigcup_{i=1}^{s} z_{m,i}]$ is the unique element in $\operatorname{span}\{f_1,..., f_{m+s-1}\}$ that verifies

$$L_j p\left[\bigcup_{i=1}^{s} z_{m,i}\right] = z_j, \qquad \forall L_j \in \bigcup_{i=1}^{s} \mathscr{L}_{m,i},$$

and analogously

$$p\left[\bigcup_{i=1}^{s} \mathscr{L}_{m,i} f_k\right].$$

From (28) we get $\alpha_n^{m+s}, \alpha_n^{m+s+1},..., \alpha_n^n$ as the unique solution of

$$\sum_{k=m+s}^{n} \alpha_n^k \left(L_j f_k - L_j p\left[\bigcup_{i=1}^{s} \mathscr{L}_{m,i} f_k\right]\right)$$
$$= z_j - L_j p\left[\bigcup_{i=1}^{s} \mathscr{L}_{m,i}\right], \qquad \forall L_j \in \mathscr{L}_n - \bigcup_{i=1}^{s} \mathscr{L}_{m,i}. \tag{29}$$

EXAMPLES. For the general case, let $m=3$, $n=6$, $s=2$, $f_1=1$, $f_2=x$, $f_3=y$, $f_4=x^2$, $f_5=xy$, $f_6=y^2$, $L_1 f = f(0,0)$,

$$L_2 f = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(0,0)}, \qquad L_3 f = \left.\frac{\partial f(x,y)}{\partial y}\right|_{(0,0)},$$

$L_4 f = f(0,1)$, $L_5 f = f(1,0)$, $L_6 f = f(1,1)$, $\mathscr{L}_{3,1} = \{L_1, L_2, L_3\}$, $\mathscr{L}_{3,2} = \{L_1, L_4, L_5\}$.

Then, one can easily obtain by direct computation

$$p_{3,1}[f]=f(0,0)+x\frac{\partial f(0,0)}{\partial x}+y\frac{\partial f(0,0)}{\partial y},$$

$$p_{3,2}[f]=f(0,0)+x(f(1,0)-f(0,0))+y(f(0,1)-f(0,0))$$

and for $X=(x,y)$, by (12)

$$\lambda_1=1-x, \qquad \lambda_2=x.$$

Fomula (11) is now

$$p_6[f]=(1-x)\,p_{3,1}[f]+xp_{3,2}[f]+\alpha_6^5xy+\alpha_6^6(y^2-xy).$$

Here one cannot apply L_5 and L_6 to both members to compute α_6^5 and α_6^6 because

$$\begin{vmatrix} L_5(xy) & L_5(y^2-xy) \\ L_6(xy) & L_6(y^2-xy) \end{vmatrix}=0.$$

However, we can use (27) with $\mathscr{L}_{3,i}=\mathscr{L}_{3,1}$ or $\mathscr{L}_{3,2}$ to obtain α_6^4, α_6^5, α_6^6.

As an example of the application of (29), let $m=1$, $n=3$, $s=2$, $f_1=1$, $f_2=x$, $f_3=y$, $L_1f=f(0,0)$, $L_2f=f(1,0)$, $L_3f=f(0,1)$. Now formula (11) is

$$p_3[f]=(1-x)f(0,0)+xf(1,0)+\alpha_{3,3}\,y$$

and (29) is

$$\alpha_{3,3}=f(0,1)-f(0,0).$$

Often the divided differences can be constructed in another simple way. As a particular case of [9, Theorem 2] we have the following result.

With the notations of Section 1 let us assume that there are $\mathscr{L}_{m,1}$, $\mathscr{L}_{m,2},\ldots,\mathscr{L}_{m,n-m+1}$ such that

(a) $\operatorname{card}\mathscr{L}_{m,i}=m$, $1\leqslant i\leqslant n-m+1$,

(b) $\det(L_jf_k)\neq 0$, $1\leqslant k\leqslant m$, $L_j\in\mathscr{L}_{m,i}$, $i=1,2,\ldots,n-m+1$,

(c) $m>\operatorname{card}(\mathscr{L}_{m,i}\cap\mathscr{L}_{m,i+1})\geqslant r\geqslant 0$, $i=1,2,\ldots,n-m$,

(d) $\bigcup_{i=1}^{n-m+1}\mathscr{L}_{m,i}=\mathscr{L}_n$.

Then the divided differences α_n^k, $m+1\leqslant k\leqslant n$ can be determined as the unique solution of the following system with $m(n-m)$ equations and $n-m$ unknowns,

$$\sum_{k=m+1}^{n}\alpha_n^k(\alpha_{m,i+1}^j[f_k]-\alpha_{m,i[fk]}^j])$$
$$=\alpha_{m,i+1}^j[z]-\alpha_{m,i}^j[z], \qquad i=1,2,\ldots,n-m;\ j=1,2,\ldots,m, \quad (30)$$

where $\alpha^j_{m,i}[f_k]$ is the foefficient of f_j in the unique solution of the problem

$$p \in \operatorname{span}\{f_1, f_2, ..., f_m\}, \qquad L_h(p) = L_h(f_k), \qquad \forall L_h \in \mathscr{L}_{m,i},$$

and analogously $\alpha^j_{m,i}[z]$ for

$$p \in \operatorname{span}\{f_1, ..., f_m\}, \qquad L_h(p) = z_h, \qquad h \in I_{m,i}.$$

If in addition, for each $i = 1, 2, ..., n-m$, there is $\mathscr{L}_{r,i}$ such that

$$\mathscr{L}_{r,i} \subset \mathscr{L}_{m,i} \cap \mathscr{L}_{m,i+1}, \tag{31}$$

$$\operatorname{card} \mathscr{L}_{r,i} = r, \tag{32}$$

$$\det(L_h f_t) \neq 0, \qquad 1 \leqslant t \leqslant r, \qquad L_h \in \mathscr{L}_{r,i}, \tag{33}$$

then system (30) can be reduced to a subsystem with the same formulation but with $i = 1, 2, ..., n-m$; $j = r+1, ..., m$.

Clearly the simplest particular case is for $r = m-1$. Then

$$\mathscr{L}_{r,i} = \mathscr{L}_{m,i} \cap \mathscr{L}_{m,i+1}, \qquad \forall\, i,$$

and we have a $(n-m) \times (n-m)$ system in (30).

Let us observe now the case $m = n-1$ in (29) and (30). In (29), if $s = 1$ and $\mathscr{L}_{n-1,1} = \{L_1, L_2, ..., L_{n-1}\}$, we have $\mathscr{L}_n - \mathscr{L}_{n-1,1} = \{L_n\}$ and

$$\alpha^n_n = \alpha^n_n[z] = \frac{z_n - L_n p_{n-1}[z]}{L_n f_n - L_n p_{n-1}[f_n]}, \tag{34}$$

where $p_{n-1}[z]$ and $p_{n-1}[f_n]$ are in span $\{f_1, ..., f_{n-1}\}$ and verify, for $1 \leqslant j \leqslant n-1$,

$$L_j p_{n-1}[f_n] = L_j f_n,$$

$$L_j p_{n-1}[z] = z_j.$$

In (30), we have

$$\alpha^n_n = \alpha^n_n[z] = \frac{\alpha^{n-1}_{n-1,2}[z] - \alpha^{n-1}_{n-1,1}[z]}{\alpha^{n-1}_{n-1,2}[f_n] - \alpha^{n-1}_{n-1,1}[f_n]}. \tag{35}$$

Equations (34) and (35) are respectively generalizations of the well-known formulas

$$f[x_0, x_1, ..., x_n] = \frac{f(x_n) - p_{n-1}(x_n)}{\prod_{i=0}^{n}(x_n - x_i)},$$

$$f[x_0, x_1, ..., x_n] = \frac{f[x_0, x_1, ..., x_{n-1}] - f[x_1, ..., x_n]}{x_0 - x_n}$$

for usual divided differences.

6. Applications: Interpolation in Six Points of R^2 by Polynomials of Degree Two

The above methods are less practical for interpolation in R^2 than that of [4] when this method can be used. However, sometimes one cannot use the method of [4] alone, but in collaboration with those of this paper, as the following example shows us.

Let us consider the interpolation problem

$$p(P_i) = f(P_i), \qquad i = 1, 2, ..., 6, \ p \in \mathscr{P}_2, \tag{36}$$

where P_i are any different points in R^2.

Clearly, the determinant of the coefficients of (36) is zero, if all the points p_i are on two straight lines r_0, r_1. Hence we only consider the following two cases:

If three points are on r_0, two points in r_1 and the other one is neither on r_0 nor on r_1, then the method of [4] can be advantageously used.

If there are not three points P_i on a straihgt line, then P_1 and P_2 determine a straight line r_0, P_3 and P_4 determine another one r_1 and P_5 and P_6, r_2. As it is shown in [4] it is impossible to get $\mathscr{P}_2$ as the interpolation space if we use that method to solve the problem. However, the method can be used to construct the unique solutions of the two problems

$$p(P_i) = f(P_i), \qquad i = 1, 2, 3, 4, 5, \quad p \in \mathscr{P}, \tag{37}$$

and

$$p(P_i) = f(P_i), \qquad i = 1, 2, 3, 4, 6, \quad p \in \mathscr{P}, \tag{38}$$

in a space $\mathscr{P}$ of dimension 5 such that $\mathscr{P}_1 \subset \mathscr{P} \subset \mathscr{P}_2$. A basis $\{f_1, f_2, f_3, f_4, f_5\}$ of $\mathscr{P}$ can be constructed easily as in [4]. With the above notations the solutions of (37) and (38) would be written as $p_{5,1}[f]$, $p_{5,2}[f]$ with $\mathscr{L}_{5,1} = \{L_1, L_2, L_3, L_4, L_5\}$ and analogously $\mathscr{L}_{5,2}$ and

$$L_i(f) = f(P_i), \qquad i = 1, 2, ..., 6.$$

Also we denote

$$\mathscr{L}_6 = \mathscr{L}_{5,1} \cup \mathscr{L}_{5,2}.$$

At first we do not know if

$$\det(L_i f_j) \neq 0, \qquad 1 \leqslant i, j \leqslant 6,$$

and so we are going to check it.

We also can easily construct $p_{5,1}[f_6]$ and $p_{5,2}[f_6]$, and then the polynomial $p_{5,1}[f_6]-p_{5,2}[f_6]$ belongs to the space $\mathscr{P}$ and vanishes at P_1, P_2, P_3, P_4. If it also vanishes at P_5 or P_6, then we shall have

$$p_{5,1}[f_6] \equiv p_{5,2}[f_6] \tag{39}$$

and hence, there exist β_i not all zero such that

$$\sum_{i=1}^{5} \beta_i f_i(P_j) - f_6(P_j) = 0, \qquad j = 1, 2, \ldots, 6,$$

and therefore

$$\det(L_i f_j) = 0, \qquad 1 \leqslant i, j \leqslant 6. \tag{40}$$

Thus (36) would have no solution in general.

Clearly (40) implies (39) and hence if (39) is not verified, then (36) has a unique solution. In this case, by Theorem 1, for any (x, y) such that

$$p_{5,1}[f_6](x, y) \neq p_{5,2}[f_6](x, y) \tag{41}$$

we have

$$p_6[f](x, y) = \lambda_1(x, y)\, p_{5,1}[f](x, y) + \lambda_2(x, y)\, p_{5,2}[f](x, y), \tag{42}$$

where

$$\lambda_1(x, y) = \frac{p_{5,2}[f_6](x, y) - f_6(x, y)}{p_{5,2}[f_6](x, y) - p_{5,1}[f_6](x, y)}, \tag{43}$$

$$\lambda_2(x, y) = \frac{f_6(x, y) - p_{5,1}[f_6](x, y)}{p_{5,2}[f_6](x, y) - p_{5,1}[f_6](x, y)}. \tag{44}$$

Also it follows from [4] that if (41) is not verified for $(x, y) \notin r_0 \cup r_1$ then we have (39) and hence (40). Therefore, if (39) is not verified we have (41) for all $(x, y) \notin r_0 \cup r_1$ and we can apply (42).

Thus, problem (36) consists of two very simple parts. At first we have to see the position of the points and how many of them are aligned. In all cases one can say inmediately if the problem has a unique solution and construct it very easily when the answer is positive.

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